THE SPATIAL FORM OF ANTIAUTOMORPHISMS OF VON NEUMANN ALGEBRAS

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1. <u>Introduction</u>. There are three problems which have been studied concerning antiautomorphisms of von Neumann algebras; the existence problem, the conjugacy problem, and their description. The latter problem includes whether they are spatial of a particular form, i.e. of the form $x \to w^* x^* w$ with w a conjugate linear isometry of a prescribed type. In the present paper we shall study the spatial problem, with main emphasis on antiautomorphisms α leaving the center elementwise on fixed, called <u>central</u> in the sequel, and with α an <u>involution</u>, i.e. $\alpha^2 = 1$. This problem with variations has previously been studied in [2,6]. E.g. it was shown in [6] that a central involution α is automatically spatial with w^2 a selfadjoint unitary operator in the center of the von Neumann algebra.

It turns out that the general problem of whether a central antiautomorphism is spatial has a solution similar to that of automorphisms, with proof also quite similar. We include these results for the sake of completeness. The main new ingredient in the paper is that if α is a central involution of the von Neumann algebra M then α is necessarily on the form $\alpha(x) = Jx^*J$ with J a conjugation, unless the commutant M' of M has a direct summand of type I_n with n odd. In the latter case it may happen that α can only be written in the form $\alpha(x)=-jx^*j$ with $j^2=-1$.

2. The results. Recall that two projections e and f in a von Neumann algebra M acting on a Hilbert space H are said to be equivalent, written $e \sim f \pmod{M}$, or just $e \sim f$, if there is a partial isometry $v \in M$ such that $v^* v = e$, $vv^* = f$. e is said to be cyclic, written $e = [M'\xi]$ if there is a vector $\xi \in H$ such that e is the projection onto the subspace spanned by vectors of the form $x'\xi$, $x' \in M'$. If w is a conjugate linear operator we denote by w^* its adjoint, viz, $(w^*\xi, \eta) = (w\eta, \xi)$. We denote by ω_ξ the positive functional $\omega_\xi(x) = (x\xi, \xi)$ on M.

Lemma. Let M be a von Neumann algebra acting on a Hilbert space H. Suppose α is a central antiautomorphism of M. Let ξ be a unit vector in H, and suppose $[M'\xi] \sim \alpha([M'\xi]) \pmod{M}$. Then we have:

(i) There exists a unit vector $\eta \in H$ such that $\omega_\eta = \omega_\xi \circ \alpha$ on M.

(ii) $[M\xi] \sim [M\eta] \pmod{M'}$.

(iii) There exists a conjugate linear partial isometry w on H such that $w^* w = [M\xi]$, $ww^* = [M\eta]$, and $w^* x w [M\xi] = \alpha(x)[M\xi]$, $x \in M$.

(iv) If $\eta = \xi$ is cyclic and $\alpha^{2n} = \iota$, the identity map, then w can be chosen so that $w^{2n} = 1$.

Proof. Let $e = [M'\xi]$ be the support of the vector state ω_ξ. Let $f = \alpha^{-1}(e)$. By assumption $e \sim f$, so there exists a partial isometry $v \in M$ such that $v^* v = e$, $vv^* = f$. Then $v\xi$ is a unit vector such that $\omega_{v\xi}(f) = (vv^* v\xi, v\xi) = (e\xi, \xi) = 1$, whence $v\xi \in f(H)$. Since

$\omega_{v\xi}(x) = \omega_\xi(v^\star xv)$, the support of $\omega_{v\xi}$ is $f = vev^\star$, hence $v\xi$ is separating for fMf. Since $\omega_\xi o\alpha$ is a normal state with support f there exists by [1,Ch. III, §1, Thm. 4] a unit vector $\eta\varepsilon f(H)$ such that $\omega_\xi o\alpha = \omega_\eta$. This proves (i).

Note that $f=vev^\star = v[M'\xi]v^\star=[M'v\xi]$. Suppose $0\neq x\varepsilon fMf$ is positive. Then $x=\alpha^{-1}(y)$ with $y\varepsilon eMe$ positive, so that $\omega_\eta(x)= \omega_\xi(y)\neq 0$. In particular, ω_η is faithful on fMf, so that its support is $[M'\eta] = [fM'f\eta] = f$. Thus $[M'\eta] \sim [M'\xi]$, and (ii) follows from [1, Ch. III, §3 Cor.]

With η as above define a conjugate linear operator $w:M\xi \to M\eta$ by $wx\xi = \alpha^{-1}(x^\star)\eta$. Then $\|wx\xi\| = \|\alpha^{-1}(x^\star)\eta\|^2 = (\alpha^{-1}(x^\star x)\eta, \eta) = (x^\star x\xi, \xi) = \|x\xi\|^2$, so that w extends to a conjugate linear isometry of $[M\xi](H)$ onto $[M\eta](H)$. Extend w to all of H by defining it to be 0 on $[M\xi](H)^\perp$. Since $w^\star w=[M\xi]$ we have for $x,y\varepsilon M$,

$$w^\star x^\star wy\xi = w^\star x^\star\alpha^{-1}(y^\star)\eta = w^\star\alpha^{-1}(y^\star\alpha(x^\star))\eta=w^\star w(y^\star\alpha(x^\star))^\star\xi=\alpha(x)y\xi .$$

Thus (iii) follows.

Finally, if $\eta=\xi$ is cyclic then w is a conjugate linear isometry such that $w^\star x^\star w=\alpha(x)$, $x\varepsilon M$. By definition of w, $w^{2k}x\xi = \alpha^{-2k}(x)\xi$, $k\varepsilon \mathbb{N}$; hence in particular, $w^{2n}x\xi = x\xi$ for all x, so that $w^{2n}= 1$.

QED.

<u>Theorem 1.</u> Let M be a von Neumann algebra and α an antiautomorphism such that $\alpha(e) \sim e$ for all projections $e\varepsilon M$. Then α is spatial.

Proof. We first note that if e' is a projection in M' then the map $\alpha_{e'}: Me' \to Me'$ defined by

$$\alpha_{e'}(xe') = \alpha(x)e'$$

is an antiautomorphism. Indeed, if $x \in M$ let c_x denote the central projection which is the intersection of all central projections q in M with $qx=x$. Since the assumption on α implies α is central, $c_x = c_{\alpha(x)}$. By [5, Lem. 3.1.1] $xe'=0$ if and only if $0=c_x c_{e'} = c_{\alpha(x)} c_{e'}$ if and only if $\alpha(x)e'=0$. Thus $\alpha_{e'}$ is well defined and injective. Since it is clearly surjective, the assertion follows.

To prove the theorem let by Zorn's lemma p' be a projection in M' maximal with respect to the property that $\alpha_{p'}$ is spatial on Mp'. Suppose $p' \neq 1$ and let $q' = 1-p'$. Let ξ be a unit vector in $q'(H)$ and let by Lemma (i) η be a unit vector in $q'(H)$ such that $\omega_\eta = \omega_\xi \circ \alpha$ on Mq'. Let $w: [M\xi](H) \to [M\eta](H)$ be as in Lemma (iii). By Lemma (ii) $[M\xi] \sim [M\eta]$ (mod M') so there is $u \in M'$ such that $u^* u = [M\eta]$, $uu^* = [M\xi]$. Then uw is a conjugate linear partial isometry which is 0 on $[M\xi](H)^{\perp}$ and isometric on $[M\xi](H)$ onto itself, such that if $x \in M[M\xi]$ then

$$(uw)^* x^* (uw) = w^* u^* xuw = w^* xw = \alpha_{[M\xi]}(x),$$

using that $u \in M'$ and $[M\xi]u = u$. Thus $\alpha_{p'} + \alpha_{[M\xi]} = \alpha_{p'+[M\xi]}$ is spatial, contradicting the maximality of p'. Thus $p'=1$, completing the proof.

Theorem 2. Let M be a von Neumann algebra with no direct summand of type II_∞ with finite commutant. Then each central antiautomorphism of M is spatial.

Proof. Let α be a central antiautomorphism of M. We may consider the different types separately. The type I portion is taken care of by [6, Lem. 4.3]. Suppose M is finite. Let Φ be the centervalued trace on M which is the identity on the center. By uniqueness of Φ, $\Phi o\alpha=\Phi$, hence $\Phi(\alpha(e))=\Phi(e)$ for all projections e. It follows that $e \sim \alpha(e)$ for all projections, hence α is spatial by Theorem 1.

Assume M is of type II_∞ with II_∞ commutant. Since the identity is the sum of central projections which are countably decomposable with respect to the center, we may assume the center is countably decomposable. By [5, Lem. 3.3.6] there is a cyclic projection $e = [M'\xi]$, ξ a unit vector, in M with central support 1 such that eq is infinite for all central projections $q \neq 0$ in M. Since α maps infinite projections onto infinite projections, $f=\alpha^{-1}(e)$ is infinite and is the support of $\omega_\xi o\alpha$. Since M' is infinite there is a unit vector η such that $\omega_\xi o\alpha=\omega_\eta$ [1, Ch. III, §8, Cor. 10].Thus $f=[M'\eta]$ is countably decomposable, and fq is infinite for all central projections $q\neq 0$, and the central support of f equals that of e since α is central. By [1,Ch. III, §8, Cor. 5] $f\sim e$. By Lemma (iii) and the maximality argument employed in the proof of Theorem 1, α is spatial.

Finally, assume M is of type III. Then each normal state is a vector state [1, Ch. III, §8, Cor. 10] so the conclusion of Lemma (i) holds. Since any two countably decomposable projections with the same central supports are equivalent in M, the argument from the II_∞ case applies to conclude that α is spatial. Q.E.D.

Remark 1. The above theorem reflects the situation for automorphisms of von Neumann algebras. For a factor M of type II_∞ with finite commutant it was shown by Kadison [4] that an automorphism is spatial if and only if it preserves the trace, or equivalently the dimension of projections. By Theorem 1 the latter condition is sufficient for an antiautomorphism α to be spatial. Conversely, if α is spatial the argument of Kadison on [4,p.324] can be repeated word by word to conclude that α preserves the dimension of projections.

The difficulty in the above situation can be avoided if α is periodic.

Theorem 3. Let M be a von Neumann algebra and α a periodic central antiautomorphism. Then α is spatial. Furthermore, if each normal state on M is a vector state (e.g. if M has a separating vector, or M' is properly infinite) then there exists a conjugate linear isometry w that $\alpha(x)=w^*x^*w$ with $w^{2n}=1$, where $2n$ is the period of α.

Proof. Let e be a projection in M. In order to show $\alpha(e)\sim e$ we may, since α is central, assume by the Comparison Theorem that $\alpha(e)\prec e$. Iterating we have $e=\alpha^{2n}(e)\prec\alpha^{2n-1}(e)\prec\ldots\prec\alpha(e)\prec e$. Thus $\alpha(e)\sim e$, and α is spatial by Theorem 1.

Now assume each normal state is a vector state. Let ψ be a unit vector. Then the state

$$\omega = \frac{1}{2n} \sum_{k=1}^{2n} \omega_\psi \circ \alpha^k$$

is a normal α-invariant state. Thus $\omega=\omega_\xi$ for a unit vector ξ, and $\omega_\xi o\alpha=\omega_\xi$. By the proof of Lemma (iv) there exists a conjugate linear partial isometry w with support and range $[M\xi]$ such that $w^{2n}=[M\xi]$, and $w^* x^* w[M\xi]=\alpha(x)[M\xi]$. A maximality argument now completes the proof.

The above theorem states that for a periodic α with M' large then w can be chosen with $w^{2n}=1$. Our last result gives a sharper statement if α an involution, Special cases of this result appeared in [6]. Recall that a conjugation is a conjugate linear isometry J such that $J^2=1$.

Theorem 4. Let M be a von Neumann algebra whose commutant has no direct summand of type I_n with n an odd integer. If α is a central involution on M then there exists a conjugation J such that $\alpha(x)=Jx^*J$, $x\varepsilon M$.

Proof. Let M act on a Hilbert space H and assume first M has no direct summand of type I. By [6, Thm. 3.7] there exist central projections p and q in M such that $\alpha|pM$ is implemented by a conjugation on $p(H)$ and $\alpha|qM$ by a conjugate isometry j with $j^2=-q$. To prove the theorem it suffices to modify j so that $\alpha|qM$ is implemented by a conjugation. We therefore assume $\alpha(x)=-jx^*j$ for $x\varepsilon M$, where $j^2=-1$. In particular α extends to an involution α of $B(H)$ implemented by j, which leaves M' globally invariant. Since M' has no direct summand of type I, neither does the fixed point algebra A of α in M' [3, 7.4.3], hence the Halving Lemma for Jordan algebras [3, 5.2.14] yields the existence of projections $e,f\varepsilon A$ with sum 1 and a symmetry $s\varepsilon A$

such that $ses=f$. Let $e_{11}=e, e_{12}=es$, $e_{21}=se=fs$, $e_{22}=f$. Then $\{e_{ij}: i,j=1,2\}$ is a set of matrix units which generates a I_2-factor M_2. Since $\alpha(e_{12})=e_{21}$, $\alpha(e_{ii})=e_{ii}$, α leaves M_2 globally invariant. Thus $B(H)=B(H_0)\otimes M_2$, and $\alpha=\alpha_1\otimes\alpha_2$ with α_1 an involution of $B(H_0)$, and $\alpha_2=\alpha|M_2$ an involution of M_2. For simplicity of notation we identify M with $M\otimes 1$, and consider M as a subalgebra of $B(H_0)$. Since an involution of a factor is implemented by a conjugate linear isometry v with $v^2=1$ or -1, [6, Thm. 3.7], it follows that $j=j_1\otimes j_2$ with $j_i^2=\pm 1$, and $\alpha|M=\alpha_1|M$ is implemented by j_1. If $j_1^2=-1$ replace j_2 by a conjugate linear isometry v with square -1, and if $j_1^2=+1$ by v with square $+1$. In either case $J=j_1\otimes v$ is a conjugation implementing α_1, and hence α on M.

It remains to consider the case when M is of type I. Since α is central we may consider the different direct summands separately, hence we may assume M is homogeneous of type I_n, $n\in \mathbb{N}\cup\{\infty\}$, with M' homogeneous of type I_r, $r\in \mathbb{N}\cup\{\infty\}$, see e.g. [1, Ch. III, §3, Prop. 2] applied to M and M'. For a Hilbert space K let t denote the transpose on $B(K)$ with respect to some orthonormal basis, and let q be the involution

$$q\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

on the complex 2x2 matrices. By [7, Thm. 2.6] M is a direct sum $M=M_1\oplus M_2$ such that α leaves each M_i invariant; $M_1=B(H_1)\otimes Z_1$, $M_2=B(H_2)\otimes B(\mathbb{C}^2)\otimes Z_2$, where in both cases Z_i is an abelian von Neumann algebra with Z_i' of type I_r. In the first case $\alpha|M_1=t\otimes\iota$, hence $\alpha|M_1$ is implemented by a conjugation, see e.g. [3, Section

7.5]. In the second case $\alpha|M_2=t\otimes q\otimes\iota$. Now q is implemented by a conjugate linear isometry j such that $j^2=-1$, while t is implemented by a conjugation J . Since by assumption M' is of type I_r with r even or $r=\infty$ there exists a conjugate linear isometry j with $j^2 = -1$, which implements a central involution on Z_2' , see [3, Section 7.5]. Thus $J\otimes j\otimes j_r$ is a conjugation which implements α on M_2 . This completes the proof of the theorem.

Remark 2. The conclusion of Theorem 4 is false if M' is of type I_n with $n\in\mathbb{N}$ odd. Let for example $M=M_m(\mathbb{C})\otimes \mathbb{C}1_n$, so that $M'=\mathbb{C}1_m\otimes M_n(\mathbb{C})$, with m even and n odd. Then there exists j on $\mathbb{C}^m$ such that $j^2=-1$, while each involution on $M_n(\mathbb{C})$ is conjugate to the transpose map. Let $\alpha(x\otimes 1)=(-jx^*j)\otimes 1_n$ on M. Then α is not implemented by a conjugation. Indeed, if J is a conjugation on $\mathbb{C}^m\otimes \mathbb{C}^n$ implementing α , then J also implements an involution on $M'=\mathbb{C}\, 1_m\otimes M_n(\mathbb{C})$, hence there would exist a conjugation J' on $\mathbb{C}^n$ such that $JxJ=-(j\otimes J')x(j\otimes J')$ for all $x\in B(\mathbb{C}^m\otimes\mathbb{C}^n)= M_m(\mathbb{C})\otimes M_n(\mathbb{C})$. Since $J^2=1$ and $(j\otimes J')^2=-1$, this is impossible by [7, Lem. 3.9], hence α is not implemented by a conjugation. This example also shows that the assumption on the normal states being vector states is necessary in Theorem 3.

References

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